Math 211 - Bonus Exercise 10 (please discuss on Forum)

1) Show that any group of order 30 has a normal subgroup isomorphic to $\mathbb{Z}/15\mathbb{Z}$.

2) Use the previous problem to classify (up to isomorphism) all groups of order 30.

3) Let $G = SL_2(\mathbb{F}_3)$, i.e. the group of 2×2 matrices with entries modulo 3 and determinant equal to 1. Show that $Z(G) \cong \mathbb{Z}/2\mathbb{Z}$ and that $G/Z(G) \cong A_4$. Hint: see last week's last bonus problem.

4) Find a Sylow p-subgroup of S_p for an odd prime number p. Then do the same for S_{2p}.

5) Find a geometric reason for why the icosahedral group is isomorphic to A_5 (forget for a second that we're in a class on abstract group theory and imagine that you're an ancient Greek mathematician contemplating the icosahedron).

6) *(Very Hard) Classification of groups of order p^3*
Let p be an odd prime. We have seen that every group of order p^2 is either isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. In this exercise we want to classify groups of order p^3.

1. Classify all abelian groups of order p^3.

Suppose now that G is not abelian.

2. Suppose that G has an element of order p^2. Show that
$$G \cong \mathbb{Z}/p^2\mathbb{Z} \rtimes_\varphi \mathbb{Z}/p\mathbb{Z}$$
for some $\varphi : \mathbb{Z}/p\mathbb{Z} \to Aut(\mathbb{Z}/p^2\mathbb{Z})$. Conclude that there is a unique non abelian group of order p^3 containing an element of order p^2.

3. Suppose now that G does not have an element of order p^2. Show that
$$G \cong (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes_\varphi \mathbb{Z}/p\mathbb{Z}$$
for some $\varphi : \mathbb{Z}/p \to Aut(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$. Conclude that there exists a unique non abelian group of order p^3 not containing an element of order p^2. Note that in that case every non-trivial element is of order p.

4. Classify all groups of order p^3.

For those interested, here is a construction of the non-abelian groups of order p^3:

5. Let G be the the group
$$G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a, b \in \mathbb{Z}/p^2\mathbb{Z} \text{ and } a \equiv 1 \pmod{p}\}$$
endowed with matrix multiplication. Show that G is a non abelian group of order p^3 isomorphic to the unique non trivial semi direct product $\mathbb{Z}/p^2\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$.

6. Let G be the Heisenberg group

$$\text{Heis}(\mathbb{Z}/p\mathbb{Z}) = \{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{Z}/p\mathbb{Z} \}$$

endowed with matrix multiplication. Show that G is a non-abelian group of order p^3 isomorphic to the unique non trivial semi direct product $(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes \mathbb{Z}/p\mathbb{Z}$.